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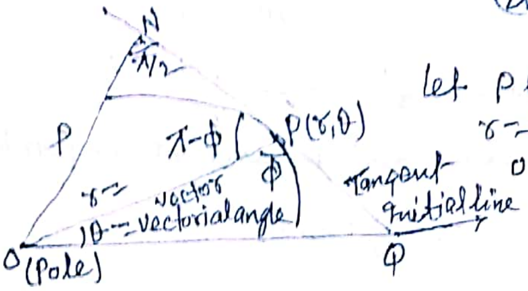
B.Sc-Part-II (H), Paper-III, Date-25-04-2020

Tangents and Normals (Differential Calculus)

Q.1 Prove that (i) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

(ii) $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

Soln.



Let P be a given point (r,θ) on the curve $r=f(\theta)$. Let ON be ⊥ from the pole O on the tangent at P. Let ON = p.

Then from the right angled ΔOPN,

$$\sin(\pi-\phi) = \frac{ON}{OP} \Rightarrow ON = OP\sin(\pi-\phi)$$
$$\Rightarrow p = r\sin\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\} \quad \left[\because \tan\phi = r\frac{d\theta}{dr}\right]$$

(i) $\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$

(ii) If $u=\frac{1}{r}$, $\therefore \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$ then, $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

Proved

Q.1 If the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at any point on it cut the axes OX and OY at P and Q respectively, prove that OP + OQ = constant

Soln The eqn of the curve is $\sqrt{x}+\sqrt{y}=\sqrt{a}$ —— I

Differentiating with respect to x, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

The eqn of the tangent to (I) at (x,y) is

$$Y-y = -\frac{\sqrt{y}}{\sqrt{x}}(X-x)$$

$$\Rightarrow \frac{Y}{\sqrt{y}} - \sqrt{y} = -\frac{X}{\sqrt{x}} + \sqrt{x}$$

$$\Rightarrow \frac{X}{\sqrt{x}} + \frac{Y}{\sqrt{y}} = \sqrt{x}+\sqrt{y} = \sqrt{a}$$

$$\Rightarrow \frac{X}{\sqrt{ax}} + \frac{Y}{\sqrt{ay}} = 1$$

$$\Rightarrow OP = \sqrt{ax} \text{ and } OQ = \sqrt{ay}$$

Hence, $OP+OQ = \sqrt{a}(\sqrt{x}+\sqrt{y})$

$= \sqrt{a}\cdot\sqrt{a} \quad [\because \sqrt{x}+\sqrt{y}=\sqrt{a}]$

$\Rightarrow OP+OQ = a$, which is constant.

Q.3. Prove that the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ touches the straight line $\frac{x}{a}+\frac{y}{b}=2$ at the point (a,b) whatever be the value of n.

Soln The equation of the curve is $\frac{x^n}{a^n}+\frac{y^n}{b^n}=2$

Differentiating with respect to x, we get

$$n\frac{x^{n-1}}{a^n}+n\frac{y^{n-1}}{b^n}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b^n}{a^n}\cdot\frac{x^{n-1}}{y^{n-1}}$$

At the point (a,b) of the curve, $\frac{dy}{dx}=-\frac{b^n}{a^n}\cdot\frac{a^{n-1}}{b^{n-1}}=-\frac{b}{a}$

Hence, the equation of the tangent to the curve at the point (a,b) is

$$y-b=\left(\frac{dy}{dx}\right)_{a,b}(x-a)=-\frac{b}{a}(x-a)$$

$$\Rightarrow \frac{y}{b}-1=-\frac{x}{a}+1 \Rightarrow \frac{x}{a}+\frac{y}{b}=2, \text{ which is independent of } n.$$

Q.4. Show that the part of the tangent to $xy=c^2$ included between the co-ordinat axes is bisected at the point of tangency.

Soln The equation of the curve is $xy=c^2$ —— (I)

Differentiating with respect to x, we get

$$1\cdot y+x\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x}$$

The equation of the tangent to (I) at (x,y) is

$$Y-y=-\frac{y}{x}(X-x) \Rightarrow xY-xy=-Xy+xy$$

$$\Rightarrow Xy+xY=2xy \Rightarrow \frac{X}{2x}+\frac{Y}{2y}=1.$$

This tangent makes intercepts $2x$ and $2y$ on the axes of x and y, that is, the tangent meets the axes of x and y at the points $(2x,0)$ and $(0,2y)$, whose middle point is (x,y), that is, the point of contact of the tangent.

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<u>Higher Algebra</u>

Q.1 To prove that the product of two determinants of the same order is itself a determinant of that order.

Proof: Let two determinants of the third order, is in general true for the product of two determinants of the nth order.

If $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

then, we have to prove that

$$\Delta_1 \Delta_2 = D$$

where,

$$D = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

It is clear that the determinant D can be expressed as the sum of $3\times3\times3$, i.e. 27 determinants of third order, three of which are

(I) $\begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 \end{vmatrix}$, i.e. $\alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{vmatrix} = 0$

(II) $\begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix}$ i.e. $\alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and (III) $\begin{vmatrix} a_1\alpha_1 & c_1\gamma_2 & b_1\beta_3 \\ a_2\alpha_1 & c_2\gamma_2 & b_2\beta_3 \\ a_3\alpha_1 & c_3\gamma_2 & b_3\beta_3 \end{vmatrix}$ i.e. $-\alpha_1\beta_3\gamma_2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Similarly, we can show that 21 out of 27 determinants will vanish and the remaining 6 determinants, Rest part of Q.1. B.Sc-part-I(H), paper-I 25-04-2020

which do not vanish, can be written as

$$\alpha_1\beta_2\gamma_3\Delta_1,\ -\alpha_1\beta_3\gamma_2\Delta_1,\ -\alpha_2\beta_1\gamma_3\Delta_1,$$

$$\therefore\ D=\Delta_1\left[\alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)-\alpha_2(\beta_1\gamma_3-\beta_3\gamma_1)+\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)\right]$$

$$=\Delta_1\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{vmatrix}=\Delta_1\Delta_2$$

Hence, the product of two determinants of the same order is a determinant of that order.

Q.2. Prove that $\begin{vmatrix}a^3 & 3a^2 & 3a & 1\\ a^2 & a^2+2a & 2a+1 & 1\\ a & 2a+1 & a+2 & 1\\ 1 & 3 & 3 & 1\end{vmatrix}=(a-1)^6$

Soln. $$\Delta=\begin{vmatrix}a^3-a^2 & 2a^2-2a & a-1 & 0\\ a^2-a & a^2-1 & a-1 & 0\\ a-1 & 2a-2 & a-1 & 0\\ 1 & 3 & 3 & 1\end{vmatrix}\quad [\because R_1\to R_1-R_2,\ R_2\to R_2-R_3,\ R_3\to R_3-R_4]$$

$$=\begin{vmatrix}a^2(a-1) & 2a(a-1) & a-1\\ a(a-1) & (a+1)(a-1) & a-1\\ a-1 & 2(a-1) & a-1\end{vmatrix}=(a-1)^3\begin{vmatrix}a^2 & 2a & 1\\ a & a+1 & 1\\ 1 & 2 & 1\end{vmatrix}$$

$$=(a-1)^3\begin{vmatrix}a^2-a & a-1 & 0\\ a-1 & a-1 & 0\\ 1 & 2 & 1\end{vmatrix}\quad [\because R_1\to R_1-R_2,\ R_2\to R_2-R_3$$

$$=(a-1)^3\begin{vmatrix}a(a-1) & a-1\\ a-1 & a-1\end{vmatrix}=(a-1)^5\begin{vmatrix}a & 1\\ 1 & 1\end{vmatrix}$$

$$=(a-1)^5.(a-1)=(a-1)^6=R.H.S$$

Proved.